N64-27313

code 1 cat. 33

NASA-CR-58048

UNPUBLISHED PRELIMINARY DATA

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35p

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TECHNICAL REPORT NO. 1
CONTRACT NASr - 94(06)
SwRI PROJECT NO. 02-1504

PREPARED FOR

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON 25, D.C.

OTS PRICE

XEROX $ 3.60 ph

MICROFILM $

JUNE 30, 1964

SOUTHWEST RESEARCH INSTITUTE
SAN ANTONIO HOUSTON

REPORTS CONTROL No.

SOUTHWEST RESEARCH INSTITUTE
8500 Culebra Road, San Antonio, Texas 78206

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ABSTRACT

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A survey of the literature pertaining to free vibrations of thin elastic shells is presented with particular attention to shells of different geometries. The abundant literature on the vibration of circular cylindrical shells is reviewed only to the extent that it sheds light on the general shell vibration problem. Only limited information concerning other shell configurations exists in the published literature. Of these the spherical shell and the truncated conical shell have received by far the most attention, however, even for these shells complete correlations between the analytic results and experimental data has yet to be made. For these and other shell geometries there is a dire need to develop new approximate techniques which enable one to solve practical problems with acceptable accuracy.

Author

INTRODUCTION

The vibrations of thin elastic shells have attracted theoretical interest among researchers in the field of mechanics and of acoustics for almost a century. The emergence of the aircraft industry and more recently, of missile and space program, added new impetus to the research efforts on this subject because of its increased practical importance. However, owing to the intrinsic complication of the problem, the analytic, as well as experimental results accumulated in the technical literature, are far from adequate to present a clear picture of the vibration problem even for the simplest shell configurations. The main difficulty lies not in the formulation of a set of equations describing the vibrations of the shell, but rather in the simplification and solution of these equations. In the existing literature, solutions with some generality of even the approximate bending theory developed by Love [1] *, (often referred to as Love's first approximation) are extremely rare.

To find escape from the nearly unsurmountable mathematical difficulties, various approximate theories are introduced to solve certain classes of shell vibration problems. The well known theory of inextensional vibrations of shells was first proposed by Lord Rayleigh

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* Numbers in brackets refer to references cited at the end of this paper.

[2] in 1881. He assumed by physical reasoning that, for the fundamental modes, the middle surface of a vibrating shell remains unstretched. The displacement functions of the middle surface can be determined with this condition, and the fundamental frequencies are then found from the potential and kinetic energy corresponding to these displacements.

Another type of shell vibration is the extensional vibration (also after Rayleigh), in which the deformation is mainly extensional. Since the flexural rigidity of a thin shell against lateral bending is always much smaller than its resistance to stretching in the middle surface, the presence of any appreciable degree of middle surface stretching will be accompanied by large membrane stresses, and consequently, will render the bending stresses relatively unimportant. For this type of vibration mode, the membrane theory naturally offers itself as an approximate approach.

From the point of view of the strain energy that is periodically stored into the shell wall during vibration, the two approximate approaches mentioned above are evidently on the opposite extremes. In the theory of inextensional vibrations, the strain energy is assumed to be associated with bending stresses only, and no membrane stresses exist since the displacements are determined by the inextension condition. On the other hand, when membrane theory is used to study the extensional vibrations of a thin shell, the strain energy is assumed to be solely associated with membrane stresses and that associated with bending is neglected. These two approximations do not contradict each other as they seem to at first sight; instead, they are complementary for the purpose of determination of various vibration modes. It is well known that, in the linear vibrations of beams or of thin plates, the lateral (or flexural) vibrations are independent of the longitudinal vibrations; in other words, the equations of motion in lateral and longitudinal directions are uncoupled and can be treated separately. In the vibrations of shells, of course, we do not have uncoupled equations of motion. However, detailed studies indicate that the vibration modes of a shell can be generally classified into distinct groups (see, for example, Refs. [8], [23], [26], [29], [30]). One of the groups is dominantly transverse vibration modes in which the strain energy associated with bending predominates, while another group is dominantly extensional vibration modes in which the strain energy is mainly due to stretching. It is because of this property of the shell vibration spectrum that the two classical approaches receive continual attention.

Aside from the two general approximate theories, there are some other approaches proposed for certain specific shell configurations. Notably, the Donnell equations for circular cylindrical shells and the Reissner equations for shallow shells are examples. In these types of theories, the governing differential equations are simplified by neglecting some unimportant terms based on geometric arguments or on an order-of-magnitude analysis. Consequently, the accuracy of their results depends largely on how well these basic assumptions are satisfied. The merit of these theories lies not only in that they yield remarkably accurate solutions under favorable conditions, but also in that they often throw light on the solutions of more elaborate theories.

It should be pointed out here that, in this review, the torsional vibration of shells of revolution have been intentionally left out of consideration. Early in 1882, Lame had found that the torsional vibrations of a spherical shell are uncoupled from other vibration modes [3]. In 1888, Love further proved that the axisymmetric torsional vibrations of a shell of revolution are independent from extensional vibration modes [4]. A more general proof was recently given by Garnet, Goldberg and Salerno [5] that the torsional vibration modes are uncoupled from both bending and extensional modes in shells of revolution.

In the following, a brief review of the numerous technical papers on vibrations of shells is presented. Although most papers fall in one of the three categories of approximate approaches mentioned above, there are a few recent papers that give complete solutions for the more exact bending theory for cylindrical and spherical shells.

VIBRATIONS OF CIRCULAR CYLINDRICAL SHELLS

Since the problem of vibrations of a circular cylindrical shell has received so much attention because of its practical importance and, perhaps more significantly, of its relative simplicity in analysis, it seems necessary to include in this survey only a brief chronological review of the most important results on this subject. The earliest study on the vibrations of cylindrical shells was made by Lord Rayleigh [2] and also treated by Love [1] . Rayleigh's inextentional theory gives two sets of vibration modes. The first set consists of two-dimensional modes (axial displacement $u = 0$), with the frequency equation

$$p_n = \frac{Eh^2}{12\rho(1-\nu^2)a^4} \frac{n^2(n^2-1)^2}{n^2+1} ,$$

where p_n is the angular frequency, n the circumferential wave number, a the radius of the middle surface, and other notations commonly used in this field. The second set of modes is three-dimensional with the frequencies given by

$$p_n^2 = \frac{Eh^2}{12\rho(1-\nu^2)a^4} \frac{n^2(n^2-1)^2}{n^2+1} \frac{1+6(1-\nu)a^2/n^2\ell^2}{1+3a^2/n^2(n^2+1)\ell^2} ,$$

where 2ℓ is the length of the cylindrical shell. It has been pointed out by Love that the inextensional displacements fail to satisfy the equations of motion and the boundary conditions and, therefore, corrections are needed for better results. The correction to the displacement required to satisfy the boundary conditions appears to be the more important one. However, the extensional strain which is necessary in order to secure satisfaction of the boundary conditions is practically confined to a very narrow region near the boundary. Furthermore, its effect in altering the total amount of the potential energy, and therefore the frequencies, is small in thin cylindrical shells.

The extensional vibrations of cylindrical shells have also been investigated fully by Rayleigh and in many other classical studies. In the case of symmetrical vibrations, the displacement takes place in planes through the axis, and the frequency equation is found to be

$$p^4 - p^2 \frac{E}{\rho(1-\nu^2)}\left(\frac{1}{a^2} + \frac{n^2\pi^2}{\ell^2}\right) + \frac{E^2 n^2 \pi^2}{\rho^2(1-\nu^2)a^2\ell^2} = 0 \quad .$$

If the length ℓ is large compared with the radius a, the two types of vibrations are: (1) almost purely radial with

$$p^2 = \frac{E}{\rho(1-\nu^2)a^2} ,$$

and (2) almost purely longitudinal, with

$$p^2 = \frac{n^2 E}{4\rho\ell^2} \quad .$$

These results are only approximate because the bending stiffness is neglected. Baron and Bleich [12] developed a method to correct and modify Rayleigh's extensional theory by the following steps. First, the natural frequencies and their corresponding mode shapes are calculated by Rayleigh's membrane theory. Then, using these mode shapes and the strain expressions of Flügge [6], they computed the maximum potential energy of bending. Finally, the corrected frequencies are obtained by employing Rayleigh's principle with the combined maximum membrane and bending energy. This procedure gives good results for lower modes and thinner shells.

Studies on the vibrations of thin cylindrical shells based on the more accurate bending theories have been made by Love, Flügge, Arnold and Warburton, and others. Love [4] worked directly with the equations of motion of his first approximation theory which leads to an asymmetric frequency determinant. Neglecting transverse normal stress and rotatory inertia, Flügge [6] derived a set of equations for freely-supported cylindrical shells, which include bending and transverse shear effects. His theory leads to a symmetrical frequency determinant which yields three natural frequencies for any particular nodal pattern, each being associated with a unique arrangement of the ratio of the displacement in the three orthogonal directions.

In their papers of 1948 and 1951, Arnold and Warburton [8], [9], presented an extensive investigation of the vibrations of cylindrical shells with freely supported ends. By using Timoshenko's strain expressions and assuming displacements similar to those assumed by Flügge, Arnold and Warburton employed Lagrange equations to set up their frequency equation. The results showed excellent agreement with their experimental data for the range of parameters tested. An interesting result revealed by their study is that the strain energy, being plotted against the number of circumferential nodes, has a minimum at certain specific number of circumferential nodes. This explains the fact that the lowest frequency does not in general associate with the simplest nodal pattern.

Many later efforts have been made at an attempt to improve the shell equations by expanding the various quantities in the three-dimensional equations of elasticity into power series of the thickness, and retaining second order terms. Kennard [10] derived a set of differential equations for cylindrical shells by this procedure. Several authors have used Kennard equations to study the axisymmetric or asymmetric motions of a cylindrical shell. Kennard's theory has been criticized by Naghdi [11] and others on the ground of inconsistency and the fact that it leads to an unsymmetric frequency determinant and thereby does not satisfy the conservation of energy.

Many other works have been directed toward a shell theory which includes the rotatory inertia and transverse-shear effects. These considerations, which are important in vibrations of thicker shells and for the higher order modes, enlarge the range of applicability of the bending theory. Mirsky and Herrmann [13] formulated a theory which includes the rotatory inertia and shear, but neglects the transverse normal stress effects. In a later paper [14] the transverse normal stress is also considered for axially symmetric cases. Their theory contains two shear constants (one associated with shear deformation in circumferential direction, the other with that in axial direction) which are determined by solving the transverse shear vibration problems in corresponding directions. Yu [15] derived a set of Donnell type equations in displacement components in which the rotatory inertia and transverse shear effects are also considered. Yu's theory gives an uncoupled eighth order equation in radial displacement and results in a cubic frequency equation. Cooper and Naghdi [16], also developed shell equations which include rotatory inertia and transverse shear stress effects. They start with Reissner's variational method and derive two sets of equations referred to as system I and system II, in which the latter is a simplified and approximate form, not including the effects of transverse shear and rotatory inertia. Greenspon [17] developed a theory for vibrations of thick-walled cylindrical shells, including all effects of transverse shear stresses, rotatory inertia, and transverse normal stresses. He includes a critical comparison between the various approximate theories and their range of applicability.

It should also be mentioned that many other papers have been published concerned with the vibration of cylindrical shells in different acoustic media or in different initial stress conditions. Fung, Sechler and Kaplan [18] studied the breathing vibrations of cylindrical shells under internal pressure. They found that the fundamental frequency increases rapidly with increasing pressure, and that the number of circumferential waves of the fundamental mode decreases when internal pressure increases. Nachbar [19] uses Donnell's equations to study the free vibrations of cylindrical shells with axial prestress. The vibrations of cylindrical shells under an initial static torque was investigated by Koval and Cranch [20] . Their experimental results show that the axial nodal lines tend to bend into helixes. Other papers treating various specific vibration problems of cylindrical shells are too many to mention here.

VIBRATIONS OF SHALLOW SHELLS AND SPHERICAL SHELLS

The inextensional vibrations of a thin spherical shell, closed at one pole and open at the other, were also first studied by Rayleigh [2] and treated briefly by Love in Reference [1]. The resulting frequency equation is much more complicated in form than that for cylindrical shells. In the case of a nearly complete sphere, the frequencies are given approximately by

$$p_n^2 = \frac{h^2}{a^4} \frac{8\mu}{3\rho} \frac{n^2(n^2-1)}{(\pi-\alpha)^2} ,$$

where p_n is the angular frequency, μ the Lame constant, α the opening angle which is nearly π for a small opening. It is seen that the frequency tends to infinity as $\alpha \rightarrow \infty$. This fact indicates that the inextensional theory does not apply to nearly complete spheres. The inextensional theory has received less attention in recent years evidently because its basic assumption restricts the solution from satisfying various boundary conditions which have significant effects on dynamic characteristics of an open spherical shell.

The extensional vibrations of a complete spherical shell have been investigated by Lamb [3] in 1883. Neglecting the bending stiffness, he found that the displacement in extensional modes can be expressed in terms of spherical harmonics of a single integral degree. An essential characteristic of this theory (also of any membrane theory) is that the frequencies of all modes are independent of the thickness. Baker [30] made an extensive study on the frequencies and natural modes of the axisymmetric extensional vibrations of a spherical shell. It is found that the frequency spectrum consists of two infinite sets: (1) the lower branch, in which all frequencies are confined in a narrow band, (2) the upper branch, in which frequencies have no upper limit. In a recent paper Kalnins [32] shows that the first set is a degenerated case of the bending modes when thickness tends to zero. Therefore, only the second set is acceptable as a part of the entire spectrum.

The first general dynamic theory of thin shells, including both flexural and extensional deformations, was given by Love [4] in 1888, but few attempts have been made to solve vibration problems of shells in a long period following Love's work. It was not until 1937 that Federhofer [34] derived three coupled equations of motion for spherical shells in displacement components, but only an approximate solution was obtained for axisymmetric vibrations of clamped shallow spherical caps. Based on Marguerre's equations, Reissner [21] reduced the governing equations to two coupled equations in transverse displacement and a stress function by neglecting longitudinal inertia terms. Using the simplified theory, Reissner [22], and Johnson and Reissner [23] [24] solved the problem of transverse vibrations of shallow spherical shells with various boundary conditions. The equations and solutions reduce to those of flat plates when the curvature on the fundamental frequency became significant even when the rise of the shell was only of the order of the thickness. Reissner's theory also indicated that, in the lateral vibrations of shallow shells, the strain energy due to stretching was as important as that due to bending. A more general study on the dynamic problem of shallow shells is advanced by Naghdi [28] which contains the effects of longitudinal inertia and is exact within the scope of bending theory of shallow shells. Kalnins and Naghdi [25] , and Kalnins [26] employed this general theory in their studies of free vibrations of a shallow spherical shell. It was found that the errors introduced by neglecting longitudinal inertia terms was very small (a few percent) if the parameters are within the admissible range of shallow shells. In a later paper, Kalnins [27] derived a set of three uncoupled differential equations for shallow spherical shells, including the effects of rotatory inertia. The frequency equation involves a five by five determinant with its elements expressed by Bessel functions. No numerical results were presented owing to the difficulty of solving this frequency equation. However, it was concluded in Kalnin's paper that, at high frequencies (in the neighborhood and above the frequency of the first thickness-shear mode), the bending theory ceased to predict natural frequencies with accuracy because the effects of thickness-shear and rotatory inertia come into play. Furthermore, in addition to the transverse modes and longitudinal modes, there exists a third set of vibration modes, the thickness-shear modes.

In summary, the vibrations of shallow shells can be treated in three stages: (1) for frequencies of the order of magnitude $w = (E/\rho)^{1/2} h/a^2$, the vibration is predominantly transverse, and Reissner's theory [21] applies, (2) for frequencies of the order of magnitude $w = (E/\rho)^{1/2}/a$, the longitudinal inertia must be considered and Naghdi's equations [28] must be employed, (3) when the frequencies are of the order of magnitude $w = (E/\rho)^{1/2}/h$, then the secondary effects of transverse-shear and rotatory inertia are no more negligible, and Kalnins' modified equations should be used. It appears that the improvement in accuracy and generality is accomplished by great sacrifice of mathematical simplicity and readiness to yield numerical results.

Based on Naghdi's general theory with longitudinal inertia neglected, Archer [29] investigated the influence of uniform initial stress states on the frequencies of transverse vibrations of shallow spherical shells. When the initial stress is a compressive one, the frequency decreases as the compressive stress gets larger, until the buckling state is reached with zero frequency.

The first successful effort in solving free-vibration problems of nonshallow spherical shells was made by Naghdi and Kalnins [31]. Based on Love's bending theory, they derived two coupled differential equations for torsionless axisymmetric motion in normal displacement and a stress function. The solution is expressed in terms of Legendre functions of the first and second kind, but the frequency equation can not be solved in explicit form owing to its transcendental character. Numerical analysis shows that, for axisymmetric vibrations, the membrane theory gives an acceptable fundamental frequency only for very small thickness. In the asymmetric case the calculation of frequencies, according to bending theory, requires in principle the evaluation of a four-by-four determinant containing associated Legendre functions of complex degrees and their derivatives. No attempts are made to obtain numerical results because of the fact that, except for the Legendre polynomials, the available tabulation of Legendre functions is incomplete and limited. In a later paper, Kalnins [32] made an extensive analysis of the effects of bending on axisymmetric vibrations of nonshallow spherical shells closed at one pole and open at the other with various conditions. The results show that the frequency spectrum consists of two infinite sets of modes, which are coupled and interspersed. According to the ratio of bending-strain-energy to total strain energy, Kalnins designated one set the bending modes and the other set membrane modes. Though the thickness shear and rotatory inertia are neglected, the computation for natural frequencies is lengthy and involved. In a recent paper, Prasad [33] advanced an analysis for vibrations of spherical shells that incorporates the effects of thickness shear and rotatory inertia besides the coupling of extensional and transverse vibrations. By a suitable choice of auxiliary variables, the governing differential equations are reduced to three; two are uncoupled partial differential equations and the remaining one is partially coupled. The choice of the auxiliary variables is similar to those used by Kalnins for shallow spherical shells [27] . The solutions are also expressed in associated Legendre functions and no numerical results are available.

As a consequence of the works due to Naghdi, Kalnins and Prasad, some general remarks can be made concerning vibrations of nonshallow spherical shells. The membrane theory and inextensional theory are valuable in predicting part of the frequency spectrum, though the extensional and lateral vibrations are always coupled. The coupling effects are especially important for the lower frequencies in the set of bending modes. The longitudinal inertia terms are in general not negligible in the vibrations of nonshallow shells with nonzero Gaussian curvature. Furthermore, the general spectrum of vibrations of a thin spherical shell, including thickness-shear and rotatory inertia effects, can be classified into five distinct infinite sets: (1) extensional modes which depend least on thickness, (2) transverse modes, (3) thickness shear modes, coming into prominence at higher frequencies, (4) rotational in-plane motions, and (5) rotational surface motion in shear.

VIBRATIONS OF CONICAL SHELLS

Although the circular conical shell has probably the simplest geometry besides the circular cylindrical shell and the spherical shell, its analysis involves much greater mathematical difficulty. An important reason is that the governing differential equations of conical shells have variable coefficients because of the change in curvature from point to point. Most work dealing with conical shells, therefore, has to rely on the Rayleigh-Ritz method or numerical integration.

The inextensional deformation of a conical shell closed at the apex has been proven impossible. Also, all thin shell theories cease to be valid near the apex because of the violation of the basic assumption that the thickness is small compared with the minimum radius of curvature. Therefore, all works concerning conical shells apply to truncated conical shells only. Some early studies on axisymmetric inextensional vibrations of conical shells, clamped at one end and free at the other, can be found in German technical literature (see Ref. [36]) and the results are reported to check poorly with experimental data. Saunders and Pasley [37] made an investigation on a particular engineering problem by applying Rayleigh's inextensional theory. They calculated the natural frequencies of a truncated conical shell with a spherical shell segment connected to the minor end of the cone. Numerical results and experimental data were provided and good agreement was observed.

In a later paper, Saunders, Wisniewski, and Paslay [38] employed the Rayleigh-Ritz method to investigate a more general case. By assuming the displacement components and using a strain energy expression including both bending and stretching energy, they calculated the natural frequencies of the asymmetric vibrations of a conical shell clamped at the minor end, hinged or free on the other edge. Their numerical results showed also that the inextensional theory does not give satisfactory solution, especially for low circumferential wave numbers.

Federhofer [35] made a general study of vibrations of conical shells with arbitrary conicity. Assuming the displacement components to be polynomial functions of the coordinate along the generator, he determined the frequencies by a Rayleigh-Ritz procedure. Though good approximation to the natural frequencies of frustums of wide vertex angle may be obtained, Federhofer's polynomial assumption cannot be expected to yield accurated results for frustrums of small vertex angles. Grigalyuk [36] made an analysis of freely-supported conical shells by an energy approach. By assuming displacement components to be trigonomitric functions in the axial direction, he obtained the equations of motion by Lagrange's equation, and solved the resulting frequency equation numerically. Some numerical values of the frequencies were tabulated for a closed conical shell. A somewhat similar assumption and procedure has been employed by Herrmann and Mirsky [39] to analyze the vibration of slightly conical shells with freely supported ends. Numerical results show that, for short shells, the conicity lowers the frequency, while for long shells, the frequency increases appreciably with conicity. It was also found that the conicity has strongest influence on frequency when the number of circumferential waves is three. Seide [40] used an energy expression of his Donnell type theory for conical shells to investigate the problem. The longitudinal inertia terms are neglected, and two types of boundary conditions are considered, namely, simple support with and without circumferential restraint. The frequency equation contains an infinite determinant which is solved numerically by obtaining the roots of truncated determinants.

Goldberg, Bogdanoff and Marcus [41] used the approach of numerical integration to calculate the frequencies and mode shapes of axisymmetric vibrations of a truncated conical shell. They converted the governing equations into a system of six first-order differential equations which were convenient to be integrated numerically. The determination of the natural frequencies was essentially achieved by trial and error. The nonvanishing value of the coefficient determinant served to indicate the error of the trial value of frequency. In a more recent paper, Goldberg, Bogdanoff and Alspaugh [42] extended this technique to the case of asymmetric vibrations of conical shells. In this case, the numerical integration has to be carried out for an eighth order system of twelve equations. Since little physical insight could be revealed by a limited number of weakly-related numerical solutions, the method developed by Goldberg, et. al., seems to be valuable primarily in solving some particular engineering problems rather than in probing the entire frequency spectrum of the conical shell with arbitrary conicity and other parameters.

Besides the theoretical efforts mentioned above, attention should be paid to a recent experimental study on vibrations of conical shells made by Watkin and Clary [43]. In their paper it was reported that conical frustums, with free-free end conditions, displayed interesting trends in that at high resonant frequencies, a greater number of circumferential nodes occur at the major end than at the minor end. This indicates that there are possible natural modes with nodal lines neither along the generator nor along the parallel circle. Similar experimental results were observed by Hoppmann, et. al., in asymmetric vibrations of shallow shells [44] and nonshallow paraboloidal shells of revolution [45]; namely, that nodal lines of asymmetric modes are not along the lines of principle curvature. It seems that analytic explanation of this phenomenon is extremely difficult.

VIBRATIONS OF SHELLS OF OTHER GEOMETRY

Besides the three types of shells discussed above, there are many other shell configurations which are of practical interest, such as, cylindrical shells with noncircular cross section, ellipsoidal shells, paraboloidal shells, complete or incomplete toroidal shells and non-shallow curved panels, etc. It is not surprising that the problem of vibrations of these shells has remained almost untouched as one recalls the analytic difficulties in the treatment of their static problems, even approximately.

In a recent paper, Hwang [48] derived a membrane theory for axisymmetric extensional vibrations of shells of revolution. From the two coupled equations of motion in displacement components, he eliminated the transverse displacement and obtained a second order differential equation in meridional displacement with variable coefficients. Numerical results were included for an ellipsoidal shell of revolution and for a hemispherical shell. It was found that the frequency spectrum consists of two sets, the lower sequence and the upper sequence, similar to what Baker [30] obtained in 1961 for spherical shells. From the previous discussion of axisymmetric extensional vibrations of spherical shells, it can be concluded that the first set is a degenerated case of the bending modes which cannot be treated satisfactorily by membrane theory unless the thickness is extremely small. In the last section of his paper, Hwang included a discussion of the edge effect solution to account for the effects of bending and transverse shear stresses near the boundary. A similar approach has been employed by DeSilva and Tersteeg [46] in treating the same problem except that the longitudinal inertia is neglected. From the results of Kalnins' paper [32], it appears that the bending effects are important for the bending modes (the lower branch in membrane theory), while the longitudinal inertia is important for the membrane modes (the upper branch in membrane theory). Therefore, the simultaneous neglect of bending effects and longitudinal inertia seems to limit the application of their theory. In the second part of their paper, DeSilva and Tersteeg also developed a boundary layer theory by assuming that the transverse displacement component predominates.

Shiraishi and DiMaggio [47] used a purturbation approach to analyze the extensional axisymmetric vibrations of prolate spheroidal shells. The thickness of the shell is not uniform since the inner and outer surfaces are formed by rotating confocal ellipses with respect to their minor axis. The solutions are obtained in power series of the eccentricity which have the solution for a spherical shell as their first term, and converge rapidly for small eccentricities.

Hoppmann, Cohen and Kunukkasseril [45] have made an attempt to use Love's bending theory in their investigation of axisymmetric vibrations of paraboloidal shells of revolution, but no solutions are obtained because of the complexity of the equations. However, experimental results are reported for frequencies and nodal patterns of axisymmetric and asymmetric vibrations of two models of paraboloidal shells of revolution.

SUMMARY

As pointed out earlier, the analysis of shell vibrations is extremely complex and involved. The difficulties in obtaining analytic solutions to dynamic shell equations are eminent. To solve the governing partial differential equations one has to confront the following mathematical tasks: to make a judicious choice of possible auxiliary variables which might uncouple the equations, to select a proper form for the eigen functions which might reduce some equations to ordinary differential equations, and finally, to perform an inordinate amount of computer work if the solutions are not expressible in the highly limited tabulated functions.

To date, most investigations have been concerned with a few types of shells of revolution with very simple geometry, even for which the solutions in asymmetric cases are still unsatisfactory to account for all experimental observed phenomena. Further analytic investigations will inevitably involve sophisticated, mathematical ingenuity, (e.g., Refs. [27], [31] and [33]). Therefore, from an engineer's view point, there is a dire need to develop new approximate techniques which enable one to solve more practical problems with acceptable accuracy. It seems that the following two approaches deserve more attention and systematic development.

1. The boundary layer theory. (e. g., Ref. [46] [48]).

In most practically important vibration modes of open shells (or composite shells) the strains in a narrow region along the edge (or connection seam) are more complicated than those in the internal part of the shell. A separate treatment of the boundary layer can considerably simplify the analysis. The internal part can usually be regarded as a region of purely inextensional vibration or of purely extensional vibration. The boundary conditions which are not satisfied by the simplified solutions may be satisfied with the help of additional boundary layer solutions.

2. Purturbation theory. (e. g., Ref. [47]). When the shell configuration deviates only slightly from a simpler one whose solution is available, the various quantities or terms of the governing differential equations can be expanded into power series of some small parameters specifying the deviation, and then the differential equations can be simplified by neglecting high order terms in such parameters, and solutions can be obtained in power series of the same. This approach may be proven invaluable for, such as, cylindrical shells with an elliptic cross section which has a small eccentricity, or for slightly bend circular cylindrical shells, etc.

In concluding this review, it is felt that for most practical important problems remaining so far unsolved or poorly solved, parallel works should be carried out in theoretical and experimental investigations. Their mutual guidance and feedback type corrections may eventually bring forth a better understanding of the dynamic characteristics of thin shells.

REFERENCES

1. Love, A. E. H., A TREATISE ON THE MATHEMATICAL THEORY OF ELASTICITY, 4th Edition, pp. 512-552, Dover, New York, (1944).

2. Lord Rayleigh, THEORY OF SOUND, 2nd Edition, Dover, New York, (1945).

3. Lamb, H., "On the Vibrations of a Spherical Shell," Proc. London Math. Soc., 14, pp. 50-56, London, (1883).

4. Love, A. E. H., "The Small Vibrations and Deformations of a Thin Elastic Shell," Phil. Trans. Roy. Soc., Ser. A 179, pp. 491-546, London, (1888).

5. Garnet, H., Goldberg, M. A. and Salerno, V. L., "Torsional Vibrations of Shells of Revolution," Journal of Applied Mechanics, 28, pp. 571-573, (1961).

6. Flügge, W., STATIK UND DYNAMIK DER SCHALEN, pp. 115, 227-232, Springer, Berlin, (1934).

7. Timoshenko, S., THE THEORY OF PLATES AND SHELLS, p. 430, McGraw-Hill, New York, (1940).

8. Arnold, R. N. and Warburton, G. B., "Flexural Vibrations of Walls of Thin Cylindrical Shells Having Freely Supported Ends," Proc. Roy. Soc., 197, Ser. A, pp. 238-256, London, (1949).

9. Arnold, R. N. and Warburton, G. B., "The Flexural Vibrations of Thin Cylinders," Journal Proc. Inst. Mech. Eng., 167, pp. 62-74, (1953).

10. Kennard, E. H., "The New Approach to Shell Theory: Circular Cylinders," Journal of Applied Mechanics, 20, pp. 33-40, (1953).

11. Naghdi, P. M., "A Survey of Recent Progress in the Theory of Elastic Shells," Applied Mechanics Review, 9, pp. 365-367, (1956).

12. Baron, M. L. and Bleich, H. H., "Table of Frequencies and Modes of Free Vibrations of Infinitely Long Thin Cylindrical Shells," Journal of Applied Mechanics, 21, pp. 178-184, (1954).

13. Mirsky, I. and Herrman, G., "Nonaxially Symmetric Motions of Cylindrical Shells," Journal of Acous. Soc. Am., 29, pp. 1116-1123, (1957).

14. Mirsky, I. and Herrman, G., "Axially Symmetric Motions of Thick Cylindrical Shells," Journal of Applied Mechanics, 25, pp. 97-102, (1958).

15. Yu, Y. Y., "Free Vibrations of Thin Cylindrical Shells Having Finite Length With Freely Supported and Clamped Edges," Journal of Applied Mechanics, 22, pp. 547-552, (1955). Also, Journal of Aerospace Sciences, 25, pp. 299-715, (1958).

16. Cooper, R. M. and Naghdi, P. M., "Propagation of Non-axially Symmetric Waves in Elastic Cylindrical Shells," Journal of Acous. Soc. Am., 29, pp. 1365-1375, (1957).

17. Greenspon, J. E., "Vibrations of a Thick-Walled Cylindrical Shell—Comparison of the Exact Theory With Approximate Theories," Journal of Acous. Soc. Am., 32, pp. 571-578, (1960).

18. Fung, Y. C., Sechler, E. E. and Kaplan, A., "On Vibrations of Thin Cylindrical Shells Under Internal Pressure," Journal of Aeronautical Sciences, 24, pp. 650-660, (1957).

19. Nachbar, W., "Characteristic Roots for Donnell's Equations With Uniform Axial Prestress," Journal of Applied Mechanics, 29, pp. 434-435, (1962).

20. Koval, L. R. and Cranch, E. T., "On the Vibrations of Thin Cylindrical Shells Subjected to an Initial Static Torque," 4th U. S. Nat'l. Congr. of Appl. Mech., (1962).

21. Reissner, E., "On Transverse Vibrations of Thin Shallow Elastic Shells," Quarterly of Applied Mathematics, 13, pp. 169-176, (1955).

22. Reissner, E., "On Axisymmetrical Vibrations of Shallow Spherical Shells," Quarterly of Applied Mathematics, 13, pp. 279-290, (1955).

23. Johnson, M. W. and Reissner, E., "On Inextensional Deformations of Shallow Elastic Shells," Journal of Math. and Phys., 34, pp. 335-346, (1956).

24. Johnson, M. W., and Reissner, E., "On Transverse Vibrations of Shallow Spherical Shells," Quarterly of Applied Mathematics, 15, pp. 367-380, (1958).

25. Kalnins, A. and Naghdi, P. M., "Axisymmetric Vibrations of Shallow Elastic Spherical Shells," Journal of Acous. Soc. Am., 32, pp. 342-347, (1960).

26. Kalnins, A., "Free Nonsymmetric Vibrations of Shallow Shells," Proc. 4th U. S. Nat'l. Congr. Appl. Mech., 1, pp. 225-233, (1962).

27. Kalnins, A., "On Vibrations of Shallow Spherical Shells," Journal of Acous. Soc. Am., 33, pp. 1102-1107, (1961).

28. Naghdi, P. M., "On the General Problem of Elastokinetics in the Theory of Shallow Shells," Proc. IUTAM Symposium on the Theory of Thin Elastic Shells, pp. 301-330, Delft, Holland, (1959).

29. Archer, R. R., "On the Influence of Uniform Stress States on the Natural Frequencies of Spherical Shells," Journal of Applied Mechanics, 29, pp. 502-505, (1962).

30. Baker, W. E., "Axisymmetric Modes of Vibrations of Thin Spherical Shells," Journal of Acous. Soc. Am., 33, pp. 1749-1758, (1961).

31. Naghdi, P. M., and Kalnins, A., "On Vibrations of Elastic Spherical Shells," Journal of Applied Mechanics, 29, pp. 65-72, (1962).

32. Kalnins, A., "Effect of Bending on Vibrations of Spherical Shells," Journal of Acous. Soc. Am., 36, pp. 74-81, (1964).

33. Prasad, C., "On Vibrations of Spherical Shells," Journal of Acous. Soc. Am., 36, pp. 489-494, (1964).

34. Federhofer, K., "Zur Berschnung der Eigenschwingurgen der Kugelschale," Sitzungsberichte der Akademie der Wissenschaften in Wien, Math. -naturwiss. Klasse, 146-2A, pp. 57-69, 505-514, (1937).

35. Federhofer, K., "Free Vibrations of Conical Shells," NASA TT F-8261, (1962). (Translated from Ingr. -Arch., 9, (1938)).

36. Grigolyuk, E. I., "Small Oscillations of Thin Resilient Conical Shells," NASA TT F-25, (1960). (Translated from Izvest. Akad. Nauk SSSR, O. T. D., No. 6, (1956)).

37. Saunders, H., and Paslay, P. R., "Inextensional Vibrations of a Sphere-Cone Shell Combination," Journal of Acous. Soc. Am., pp. 579-583, (1959).

38. Saunders, H., Wisniewski, E. J., and Paslay, P. R., "Vibrations of Conical Shells," Journal of Acous. Soc. Am., 32, pp. 765-772, (1960).

39. Herrmann, G., and Mirsky, I., "On Vibrations of Conical Shells," Journal of Aerospace Sciences, 25, pp. 451-458, (1958).

40. Seide, P., "On the Free Vibrations of Simply Supported Truncated Conical Shells," Conference on Shell Theory and Analysis, Lockheed Missiles and Space Co., Research Laboratory, (1963).

41. Goldberg, J. E., Bogdanoff, J. L. and Marcus, L., "On the Calculation of the Axisymmetric Modes and Frequencies of Conical Shells," Journal of Acous. Soc. Am., 32, pp. 738-742, (1960).

42. Goldberg, J. E., Bogdanoff, J. L. and Alspaugh, D. W., "On the Calculation of the Modes and Frequencies of Vibration of Pressurized Conical Shells," AIAA 5th Annual Structural and Materials Conference, Palm Springs, California, pp. 243-249, (1964).

43. Watkins, J. D. and Clary, R. R., "Vibrational Characteristics of Thin-Walled Conical Frustum Shells, AIAA Aerospace Science Meeting," Preprint No. 64-78, New York, (1964).

44. Hoppmann, W. H., II, and Baronet, C. N., "A Study of the Vibrations of Shallow Spherical Shells," Journal of Applied Mechanics, 30, pp. 329-334, (1963).

45. Hoppmann, W. H., II, Cohen, M. I., and Kunukkasseril, V. X., "Elastic Vibrations of Paraboloidal Shells of Revolution," Journal of Acous. Soc. Am., 36, pp. 349-353, (1964).

46. DeSilva, C. N. and Tersteez, G. E., "Axisymmetric Vibrations of Thin Elastic Shells," Journal of Acous. Soc. Am., 36, pp. 666-672, (1964).

47. Shiraishi, N. and DiMaggio, F. L., "Perturbation Solution for Axisymmetric Vibrations of Prolated Spherical Shells," Journal of Acous. Soc. Am., 34, pp. 1725-1731, (1962).

48. Hwang, C., "Extensional Vibration of Axisymmetrical Shells," AIAA 5th Annual Structural and Materials Conference, Palm Springs, California, pp. 227-234, (1964).